from thermal stress and/or ablates (region 5 in Fig. 1d). The discharge is otherwise essentially the same as in the unconfined channel.

The random sparking that occurs at the cathode surface in the absence of the applied magnetic field is most likely due to sputtering associated with the bombardment of the cathode by energetic ions. Sputtering erodes the copper cathode and would account for the green coloration, characteristic of copper, which is observed in the cathode wake. The applied magnetic field diminishes the frequency and intensity of the random sparking on the cathode and the luminosity of the copper-green wake that trails from the cathode and causes both effects to disappear at fields of more than 500 gauss. Erosion of the cathode also effectively stops at 500 gauss. A possible reason for this is that the number of energetic ions that strike a given spot of the cathode surface is reduced with increasing magnetic field.

Since the maximum energy of the ions in these experiments cannot be much in excess of 25 to 100 ev, the random multiple sparks probably do not arise from the collision of single, very high energy ions with the cathode surface but are probably the result of a local collapse of positive space charge column(s) on the cathode. Each collapsing column consists of many ions of moderate energy (from a fraction of an electron volt to several electron volts). The combined energy of these ions hitting the same spot in quick succession drives a thermal spike into the cathode surface at the point of impact (the base of the column). Many of these thermal spikes possess sufficient energy to cause local sputtering of the copper cathode, and the multiple jets are visible as multiple sparks distributed at random over the cathode surface. The sputtered copper causes the green luminosity of the cathode wake. Then one of the possible mechanisms causing the suppression and eventual disappearance of the sparks and the green luminosity with increasing B is the bending of the trajectories of the moderate energy ions or ion clusters by the applied magnetic field. In that case, the ions at the top of the collapsing column will strike a different spot on the cathode from the ions at the bottom of the column and will effectively smear out the energy input to the surface so that the thermal spike is attenuated. For example, if a 10^{-3}-m-long column of monoenergetic ions (of the order of 10^4 ions under the conditions of the experiment) with a velocity of 3×10^3 m/sec (corresponding to 2 ev or the drift velocity, which results when an electric field of 150 v/m and a transverse magnetic field of 0.05 weber/m^2 are acting on the ions) collapses in the presence of a magnetic field of 0.05 weber/m^2, it will spread out on the surface of the cathode over a length of the order of $\Delta x \approx (10^{-3})^2/2r_c$ m, where r_c is the cyclotron radius given by $r_c = 4.18 \times 10^{-7}\,(U/B) = 2.5 \times 10^{-2}$ m, i.e., over $\Delta x = 2 \times 10^{-5}$ m, or roughly over 10^4 atoms of copper. Note that under these conditions the ions that are more than 5×10^{-2} m away from the cathode can never even reach it without collisions. Collisions, however, would tend to reduce their energy amd make their impacts even less effective. This computation is not meant to be quantitative; nevertheless, it indicates a possible mechanism that could account for the disappearance of the random sparks and the copper-green wake. Note also that, if the sparks were the result of impacts of single high energy ions on the cathode surface, the slight deflection of their trajectories by the magnetic field would have no effect on the frequency and intensity of the sparks.

Other possible explanations for these phenomena are that 1) the magnetic field causes the cathode spot(s) to move rapidly on the cathode surface, thereby spreading out the energy input and broadening the thermal spikes, and/or 2) the magnetic field causes a decrease of the energy of the incident ions.

As the magnetic field strength increases, the luminous region that is initially concentrated at the trailing edge of the anode (region 3, Fig. 1b) spreads upstream on the top surface and down the trailing side of the anode. The spreading of the highly luminous sheath over the surface of the anode is associated with the increased impedance of the plasma in the direction of the applied electric field (perpendicular to the applied magnetic field) which increases the cross-sectional area of the discharge and the applied voltage (for constant current). The increased applied voltage also contributes to the spreading of the discharge upstream on the anode surface, i.e., toward the accelerator inlet where the back emf UB is small. This increase of the impedance across the electrode gap enhances the tendency of the discharge to fringe outside the gap and outside the region of influence of the applied magnetic field.

The vertical deflection of the exhaust plume mentioned previously which increases with increasing magnetic field is produced by the interaction of the applied magnetic field with the axial component of the electric current. The axial component of the current, i.e., the current perpendicular to both the applied magnetic field and the applied electric field, is a Hall current that may be interpreted in terms of the tensor conductivity of a plasma in the presence of a magnetic field. The vertical deflection of the jet can be avoided by moving the anode upstream of the cathode so as to provide an axial component of electric field which suppresses the axial current.

The observations reported here were made visually and photographically and therefore are descriptions of the intensity of visible light radiated from various points in the plasma flow. From the nature of the patterns described, it seems reasonable to assume that the regions of high luminosity correspond to those of high current density and the associated high rate of ohmic heating. Experimental investigations to determine the current distribution more precisely through the use of spectroscopic, microwave, and induction-coil probing techniques are currently in progress at the Northrop Plasma Laboratories.

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A Class of Linear Magnetohydrodynamic Flows

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Introduction

THE continuity and momentum equations for the steady laminar flow of a conducting fluid of constant properties, that is, constant fluid density ρ, viscosity μ, and electrical conductivity σ, are

$$\nabla \cdot V = 0 \tag{1}$$

$$\rho(V \cdot \nabla)V + \nabla P = J \times B + \mu \nabla^2 V \tag{2}$$

where V is the velocity vector, with components u, v, and w in

Received by ARS October 31, 1962.

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the x, y, and z directions, respectively; P is the pressure; J is the electric current density vector; and B is the magnetic flux density vector. The mks system of units will be used throughout.

The Maxwell equations are

$$\nabla \cdot B = 0 \quad (3)$$

$$\nabla \times B = \mu_0 J \quad (4)$$

$$\nabla \cdot E = 0 \quad (5)$$

$$\nabla \times E = 0 \quad (6)$$

where E is the electric field density vector and μ_0 is the permeability constant.

The current conservation equation, in the absence of polarization, is

$$\nabla \cdot J = 0 \quad (7)$$

and Ohm's law, neglecting Hall effects, is

$$J = \sigma(E + V \times B) \quad (8)$$

Substituting Eq. (4) in Eq. (2), and noting the vector identity

$$(\nabla \times B) \times B = (B \cdot \nabla)B - \nabla(B^2/2) \quad (9)$$

Eq. (2) becomes

$$\rho(V \cdot \nabla)V + \nabla(P + B^2/2\mu_0) = (1/\mu_0)(B \cdot \nabla)B + \mu \nabla^2 V \quad (10)$$

Eliminating J between Eqs. (4) and (8), taking the curl of the resulting expression, and noting Eq. (6), one obtains

$$(1/\mu_0\sigma)\nabla \times (\nabla \times B) = \nabla \times (V \times B) \quad (11)$$

Expanding both sides of Eq. (11), using vector identities,

$$(1/\mu_0\sigma)[\nabla(\nabla \cdot B) - \nabla^2 B] = (B \cdot \nabla)V - (V \cdot \nabla)B + B(\nabla \cdot V) - V(\nabla \cdot B) \quad (12)$$

In view of Eqs. (1) and (3), Eq. (12) reduces to

$$(V \cdot \nabla)B = (1/\mu_0\sigma)\nabla^2 B + (B \cdot \nabla)V \quad (13)$$

Eqs. (1, 3, 10, and 13) are the pertinent equations involving V, B, and P only. However, these equations constitute a set of nonlinear differential equations.

The various nonlinear terms are

$$(V \cdot \nabla)V \qquad (B \cdot \nabla)B \qquad \nabla B^2$$
$$(V \cdot \nabla)B \qquad (B \cdot \nabla)V$$

The purpose of this discussion is to present a class of problems for which Eqs. (1) and (3) are identically satisfied, and the forementioned nonlinear terms become linear to the point where the remaining pertinent equations are easily solvable. Two sets of conditions give rise to this class of linear problems, and they are discussed below as cases 1 and 2.

Case 1

In this case

$$\begin{aligned} u &= u(y, z) & B_x &= B_x(y, z) \\ v &= w = \partial/\partial x = 0 & B_y &= 0 \\ & & B_z &= B_0 \end{aligned} \quad (14)$$

An exception to the condition $\partial/\partial x = 0$ will be made in the case of the pressure. In other words, since the flow is in the x direction, the pressure P, and only P, will be assumed to vary in the x direction. The externally applied magnetic field B_0 acts along the z direction and is assumed constant.

With this set of conditions, one concludes that

$$\nabla \cdot V = 0$$
$$\nabla \cdot B = 0$$
$$(V \cdot \nabla)V = 0$$
$$(V \cdot \nabla)B = 0$$

and

$$\nabla B^2 = \begin{cases} \partial B_x^2/\partial y \\ \partial B_x^2/\partial z \end{cases}$$
$$(B \cdot \nabla)B = B_0(\partial B_x/\partial z)$$
$$(B \cdot \nabla)V = B_0(\partial u/\partial z)$$

Eq. (10) can now be written as

$$\partial P/\partial x = (B_0/\mu_0)(\partial B_x/\partial z) + \mu \nabla_1^2 u \quad (15a)$$

$$(\partial P/\partial y) + (1/\mu_0)(\partial B_x^2/\partial y) = 0 \quad (15b)$$

$$(\partial P/\partial z) + (1/\mu_0)(\partial B_x^2/\partial z) = 0 \quad (15c)$$

Examination of Eqs. (15) shows that the pressure P must assume the form

$$P = Ax + p(y, z) \quad (16)$$

where A is a constant, and p is some function of y and z. Substituting in Eqs. (15b) and (15c) and integrating gives

$$p = -(B_x^2/2\mu_0) + C \quad (17)$$

where C is a constant. Eq. (15a) becomes

$$(B_0/\mu_0)(\partial B_x/\partial z) + \mu \nabla_1^2 u = A \quad (18)$$

In Eq. (13), the only equation that survives is

$$(1/\mu_0\sigma)\nabla_1^2 B_x + B_0(\partial u/\partial z) = 0 \quad (19)$$

where

$$\nabla_1^2 = (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$$

Eqs. (18) and (19), which are a pair of linear partial differential equations, along with the proper boundary conditions, constitute the pertinent equations for any problem for which the conditions of Eqs. (14) are valid. Once B_x and u are determined, Eqs. (16) and (17) then can be used to determine the pressure.

Case 2

In this case

$$\begin{aligned} u &= u(z) & B_x &= B_x(z) \\ v &= v(z) & B_y &= B_y(z) \\ w &= \partial/\partial x = \partial/\partial y = 0 & B_z &= B_0 \end{aligned} \quad (20)$$

Here again, an exception to the condition $\partial/\partial x = \partial/\partial y = 0$ will be made in the case of the pressure. The externally applied magnetic field B_0 acts along the z direction and is assumed constant.

With this set of conditions, one concludes that

$$\nabla \cdot V = 0$$
$$\nabla \cdot B = 0$$
$$(V \cdot \nabla)V = 0$$
$$(V \cdot \nabla)B = 0$$

and

$$\nabla B^2 = (dB_x^2/dz) + (dB_y^2/dz)$$
$$(B \cdot \nabla)B = \begin{cases} B_0(dB_x/dz) \\ B_0(dB_y/dz) \end{cases}$$
$$(B \cdot \nabla)V = \begin{cases} B_0(du/dz) \\ B_0(dv/dz) \end{cases}$$

With these, Eq. (10) becomes

$$\frac{dP}{dx} = \frac{B_0}{\mu_0}\frac{dB_x}{dz} + \mu \frac{d^2u}{dz^2} \quad (21a)$$

$$\frac{dP}{dy} = \frac{B_0}{\mu_0}\frac{dB_y}{dz} + \mu \frac{d^2v}{dz^2} \quad (21b)$$

$$\frac{dP}{dz} + \frac{1}{2\mu_0}\frac{d}{dz}(B_x^2 + B_y^2) = 0 \quad (21c)$$

Examination of Eqs. (21) shows that the pressure P must be expressed in the form

$$P = ax + by + p(z) \quad (22)$$

where a and b are constants, and $p(z)$ is some function of z. Substituting in Eq. (21c) and integrating gives

$$p = -(1/2\mu_0)(B_x^2 + B_y^2) + C \quad (23)$$

where C is a constant. Eqs. (21a) and (21b) now become

$$\frac{B_0}{\mu_0}\frac{dB_x}{dz} + \mu\frac{d^2u}{dz^2} = a \quad (24a)$$

$$\frac{B_0}{\mu_0}\frac{dB_y}{dz} + \mu\frac{d^2v}{dz^2} = b \quad (24b)$$

In addition, two of Eqs. (13) survive:

$$\frac{1}{\mu_0\sigma}\frac{d^2B_x}{dz} + B_0\frac{du}{dz} = 0 \quad (25a)$$

$$\frac{1}{\mu_0\sigma}\frac{d^2B_y}{dz^2} + B_0\frac{dv}{dz} = 0 \quad (25b)$$

Eqs. (24a) and (25a), which involve u and B_x only, may be combined to give

$$\frac{d^3u}{dz^3} - \frac{\sigma B_0^2}{\mu}\frac{du}{dz} = 0 \quad (26)$$

and

$$\frac{d^3B_x}{dz^3} - \frac{\sigma B_0^2}{\mu}\frac{dB_x}{dz} = -\frac{\mu_0\sigma B_0}{\mu}a \quad (27)$$

These two equations can be solved separately for u and B_x, respectively. Since Eqs. (24b) and (25b), which involve v and B_y, have a structure similar to that of Eqs. (24a) and (25a), it is evident that v and B_y have the same functional form, except for a constant, as u and B_x, respectively. Once B_x and B_y are determined, the pressure P can be found from Eqs. (22) and (23).

The solutions to a few problems of the two cases discussed in the foregoing are reported in the literature. Shercliff[1] obtained a complete solution for a problem of the case 1 type, namely, the flow through a straight pipe with rectangular cross section. Sherman and Sutton[2] solved a problem of the case 2 type except for the inclusion of the Hall effects in Eq. (8). The flow between parallel plates was studied by Hartmann[3] for the case of stationary plates and by Lehnert[4] for the case of relative motion between the plates. Both of these problems come under case 2 but with the additional simplifying condition that $v = 0$.

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Existence of Periodic Solutions Passing Near Both Masses of the Restricted Three-Body Problem

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The following new result is announced and an outline of its proof indicated. There exist in the restricted three-body problem with small mass ratio one-parametric families of synodically closed solution curves, which are near rotating Keplerian ellipses with arbitrary rational sidereal frequencies and appropriate positive eccentricities. By suitable selection of the parameter values, these periodic solutions can be made to come close to both attracting bodies. Thus, besides its meaning for astronomy or atomic physics possibly, the practical significance of this result for astronautics becomes apparent if one considers operating spacecraft along such paths in the Earth-sun or moon-Earth systems. (The detailed mathematical existence proof is to appear elsewhere.)

In the restricted three-body problem, one considers the motion of a particle P of negligible mass moving subject to the Newtonian attraction from two other bodies E and M, which rotate in circles about their center of gravity S. One can choose the units of time, mass, and distance such that E and M have masses $1 - \mu$ and μ, distance 1, and angular velocity 1 ($0 \leqq \mu \leqq 1$). Considering the case when P moves in the plane of E and M only and using complex position vectors drawn from S as origin, the equations of motion for P with position vector $x = x_1 + ix_2 = x(t)$ in a co-system rotating with E and M are ($\dot{} = d/dt$)

$$\ddot{x} + 2i\dot{x} - x = -(1-\mu)(x+\mu)|x+\mu|^{-3} - \mu(x+\mu-1)|x+\mu-1|^{-3} \quad [1]$$

since $-\mu e^{it}$, $(1-\mu)e^{it}$, and $z = xe^{it} = z(t)$ are the inertial position vectors of E, M, and P at time t.

For $\mu = 0$, the solutions of [1] are well known. They correspond to Keplerian motions $z(t)$, i.e., solutions of $\ddot{z} = -z|z|^{-3}$. If $z(t)$ describes an elliptic motion, its period $T_0 = 2\pi|a^{3/2}|$ is determined by the major half-axis $a > 0$ alone. In order that the corresponding $x(t)$ be periodic, it is necessary and sufficient that T_0 be commensurable with the period of M; i.e., $a^{3/2} = m/k$ with natural m and integer k, which is chosen positive respectively negative, if $z(t)$ is direct respectively retrograde. The initial conditions

$$x(0) = a(1+\epsilon) = \xi^*$$
$$\dot{x}(0) = i(c^* - \xi^{*2})/\xi^* = i\eta^* \qquad c^{*2} = a(1-\epsilon^2) \quad [2]$$

yield such a solution of [1] with $\mu = 0$ describing motion along a rotating ellipse of eccentricity $\epsilon (0 < \epsilon < 1)$ and period T_0. The synodical period of this solution is $T^* = 2\pi m = |k|T_0$, and it closes after $k - m$ revolutions around the origin, which is a focus of $z(t)$. This solution is denoted by $x^*(t)$ from here on.

Now the following result can be stated: There exist periodic solutions $x(t)$ of [1] for small $\mu > 0$ which are near the generating solutions $x^*(t)$ belonging to arbitrary k, m, and properly restricted ϵ. These solutions and their synodical periods T are continuous in μ and transfer into $x^*(t)$ for $\mu = 0$.

Presented at the ARS 17th Annual Meeting, Los Angeles, Calif., November 13–18, 1962.

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